

The role of Dirac equations in the classical mechanics of the relativistic top

Enrico Santamato*

*Dipartimento di Scienze Fisiche,
Università di Napoli "Federico II"
via Cintia, 80127 Napoli, Italy*

(Dated: August 25, 2008)

Abstract

We show that the square of the spin $1/2$ Dirac equation may occur also in the framework of the classical (i.e. non quantum) mechanics of the relativistic top. In particular, it is shown that the spin $1/2$ quantum states correspond to particular bundles of extremal curves of a suitable Lagrangian in the top configuration space. Along the extremal curves a volume measure is transported that reduces to the scalar space curvature when the extremal curves are taken as time-lines. The theory is carried out in the framework of the special relativity, although the extension to arbitrary space-time metric tensor is straightforward. A corollary of the present approach is proving the conformal invariance of Dirac's equation.

PACS numbers: 03.65.Ta;45.20.Jj

Keywords: relativistic top, quantum spin

Dirac's equation for the spin 1/2 relativistic particle is one of the cornerstones of quantum mechanics. It is commonly believed that spin 1/2 is a peculiar feature of the quantum world and that any attempt to find a classical system behaving as the spin 1/2 quantum particle is almost hopeless. In this work, it is shown that Dirac's equation (and hence the spin 1/2 quantum particle) may have room in the classical mechanics of the relativistic top, thus paving the way to possible alternative interpretations of the spin 1/2 quantum states in terms of classical objects. We start considering the simplest (and oldest) model for the relativistic spinning particle, namely the top described by six Euler angles, as made, for example, by Frenkel [1] and Thomas [2]. More precisely, we imagine that the particle follows a path $x^\mu = x^\mu(\sigma)$ in space-time, where σ is an arbitrary parameter along the path, and that it carries along with itself a moving fourleg $e_a^\mu = e_a^\mu(\sigma)$ ($a = 0, \dots, 3$). The fourleg vectors e_a^μ are normalized according to $g_{\mu\nu} e_a^\mu e_b^\nu = g_{ab}$ where $g_{\mu\nu} = g_{ab} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric tensor. Setting $\Lambda(\sigma) = \{e_a^\mu(\sigma)\}$ and $G = \{g_{\mu\nu}\}$, the normalization relations can be written in the matrix form as $\Lambda^T G \Lambda = G$, showing that the 4×4 matrix $\Lambda(\sigma)$ is a Lorentz matrix. The derivative of $e_a^\mu(\sigma)$ with respect to σ can be written as $de_a^\mu/d\sigma = \omega_\nu^\mu e_a^\nu$. The contravariant tensor $\omega^{\mu\nu} = \omega_\sigma^\mu g^{\sigma\nu}$ is skewsymmetric and can be considered as the "angular velocity" of the top in space-time. The free Lagrangian of this minimal relativistic top is

$$L_0 = mc \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} + a^2 g_{\mu\nu} g^{ab} \frac{de_a^\mu}{d\sigma} \frac{de_b^\nu}{d\sigma}} = mc \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} + a^2 d\omega_{\mu\nu} d\omega^{\mu\nu}}, \quad (1)$$

where m is the particle mass, c is the speed of light, and a is a constant having the dimension of a length. The square root in Eq. (1) ensures that L_0 is parameter invariant. In the presence of an external electromagnetic field, the total Lagrangian becomes $L = L_0 + L_{em}$, where the electromagnetic interaction Lagrangian is taken as

$$L_{em} = -\frac{e}{c} A_\mu \frac{dx^\mu}{d\sigma} - \frac{e}{c} a^2 F_{\mu\nu} \frac{ds^{\mu\nu}}{d\sigma} \quad (2)$$

where e is the particle charge and $F_{\mu\nu}$ is given by $F_{\mu\nu} = \partial A_\nu / \partial x^\mu - \partial A_\mu / \partial x^\nu$ with four-potential A_μ given by $A_\mu = (-\phi, \mathbf{A})$, ϕ and \mathbf{A} being the scalar and vector electromagnetic potentials, respectively. Notice that the only free parameter in the Lagrangian L is the scale length fixed by the constant a . For a quantum particle of mass m we expect a to be of the order of the particle Compton wavelength. The configuration space of the top described by the Lagrangian L is the principal fiber bundle whose base is the Minkowski space-time \mathcal{M}_4 and whose fiber is $SO(3, 1)$, conceived as a proper Lorentz frame manifold.

The dynamical invariance group is the whole Poincaré group of the inhomogeneous Lorentz transformations. The fourleg components e_a^μ (and the $SO(3,1)$ group) can be parametrized by six “Euler angles” θ^α ($\alpha = 1, \dots, 6$), so that the configuration space spanned by the space-time coordinates and the Euler angles is ten dimensional. When $\omega^{\mu\nu}$ is written in terms of the angles θ^α and their derivatives, the free-particle Lagrangian L_0 assumes the standard form

$$L_0 = -mc \frac{ds}{d\sigma} = -mc \sqrt{-g_{ij} \frac{dq^i}{d\sigma} \frac{dq^j}{d\sigma}}, \quad (3)$$

where $q^i = \{x^\mu, \theta^\alpha\}$ ($i = 0, \dots, 9$) are the ten coordinates spanning the dynamical configuration space of the top. Similarly, the electromagnetic interaction Lagrangian L_{em} assumes the form $L_{em} = -(e/c) A_i dq^i/d\sigma$, where $A_i = (A_\mu, A_\alpha)$ is a covariant vector. The last six entries A_α of A_i are linear combinations of the components of the magnetic and electric fields $\mathbf{H}(x)$ and $\mathbf{E}(x)$, respectively¹. The quantities g_{ij} in Eq. (3) define the metric of the top configuration space, which is so converted into the 10-D Riemann space $V_{10} = \mathcal{M}_4 \times SO(3,1)$. The extremal curves of L are the geodesics of this space. The metric tensor g_{ij} in V_{10} has the diagonal block form $g_{ij} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{\alpha\beta} \end{pmatrix}$, where $g_{\mu\nu}$ is the Minkowski metric and $g_{\alpha\beta}$ is the metric of the parameter space of Lorentz group with signature $(+, +, +, -, -, -)$ and we assumed the Euler angles θ^α ordered so that the first three angles θ^α for $\alpha = 1, 2, 3$ are associated to space rotations, and the last three angles θ^α for $\alpha = 4, 5, 6$ to Lorentz boosts. The classical mechanics induced by the extremals of the Lagrangian L on the space V_{10} is well known and it can be found also in textbooks [3, 4]. Here is enough noticing that neither the time-like vector e_0^μ of the moving fourleg is identified with the particle four-velocity $u^\mu = dx^\mu/d\tau$, $d\tau = \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$ being the proper time, nor Weysenhoff’s kinematical constraints $d\omega_{\mu\nu} u^\nu = 0$ are imposed, in general. As a consequence, the so called “center-of-mass” space-time trajectory $x^\mu(\tau)$ and the so called “center-of-energy” space-time trajectory $y^\mu(\tau)$ of our top (obtained from $dy^\mu/d\tau = e_0^\mu$) are different [see Ref. 4, chap. 20]. The main advantage of using a top described by six Euler angles is that the usual methods of analytical mechanics can be applied without worrying about kinematical constraints; but the usual picture of spin as the coadjoint action of the little Poincaré group on the particle momentum space [see Ref. 3, Chap. 3, sec. 13] is

¹ More specifically, $A_\alpha = a \xi_\alpha^a(\theta) F_a(x)$ ($\alpha, a = 1, \dots, 6$), where $F_a(x) = \{\mathbf{H}(x), \mathbf{E}(x)\}$ and $\xi_\alpha^a(\theta)$ are the Killing vectors of the Lorentz group $SO(3,1)$.

generally lost.

As we shall see soon later, little change of the Lagrangian L yields a new Lagrangian \bar{L} , grasping the essential features of the quantum spin and leading to an equation which is substantially equivalent to the square of Dirac's equation. For this reason, we will refer conveniently to the Lagrangian \bar{L} as to the “quantum” Lagrangian, but it should be clear that \bar{L} itself and all equations henceforth derived from \bar{L} have nothing quantum in nature. The Lagrangian \bar{L} is obtained from L by replacing in L_0 the metric tensor g_{ij} with the new metric tensor given by $\bar{g}_{ij} = \chi^{-2}g_{ij}$, where $\chi(q)$ is a dimensionless Weyl factor depending on all coordinates q^i , and also replacing the mass m with the scalar Riemann curvature $\bar{R}(q)$ as calculated from the metric \bar{g}_{ij} . The electromagnetic part L_{em} of \bar{L} remains unchanged as given by Eq. (2). In other words, we consider a new relativistic top described by the Lagrangian given by $\bar{L} = \bar{L}_0 + L_{em}$, where

$$\bar{L}_0 = -\hbar \frac{d\bar{s}}{d\sigma} = -\hbar \sqrt{-\gamma^2 \bar{R} \bar{g}_{ij} \frac{dq^i}{d\sigma} \frac{dq^j}{d\sigma}}, \quad (4)$$

where γ is a numeric constant to be fixed later. It is worth noting that no particle mass appear explicitly in Eq. (4) and that the overall factor \hbar was inserted for dimensional reasons only. From the Lagrangian \bar{L}_0 we see that the scalar curvature \bar{R} acts as a scalar potential on the top, and, because \bar{R} depends on χ and its derivatives, the Weyl factor χ acts on the top as a sort of pre-potential. The paths followed by the top in the configuration space $V_{10} = \mathcal{M}_4 \times SO(3, 1)$ are assumed to be the extremal curves of the action integral $\int \bar{L} d\sigma$. Of particular importance will be the bundles of extremals belonging to a family of equidistant hypersurfaces $S = \text{const.}$ in the configuration space. These bundles are obtained from the solutions of the Hamilton-Jacobi equation associated to \bar{L}

$$\bar{g}^{ij} \left(\frac{\partial S}{\partial q^i} - \frac{e}{c} A_i \right) \left(\frac{\partial S}{\partial q^j} - \frac{e}{c} A_j \right) = \bar{g}^{ij} \left(\bar{\nabla}_i S - \frac{e}{c} A_i \right) \left(\bar{\nabla}_j S - \frac{e}{c} A_j \right) = -\hbar^2 \gamma^2 \bar{R} \quad (5)$$

by integrating the differential equations $dq^i/d\sigma = \bar{g}^{ij}[\partial S/\partial q^j - (e/c)A_j]$ [5]. Moreover, we may also assume that the action function S obeys the auxiliary condition

$$\bar{\nabla}_k \left(\bar{\nabla}^k - \frac{e}{c} A^k \right) S = 0, \quad (6)$$

where in Eqs. (5) and (6) $\bar{\nabla}_i$ denotes the covariant derivative calculated from the Cristoffel symbols $\overline{\{^i_{jk}\}}$ built up from the metric \bar{g}_{ij} and the indices have been raised using this metric.

Equation (6) states the conservation of the current density $j^i = \bar{g}^{ij}[\partial S/\partial q^j - (e/c)A_j]\sqrt{\bar{g}}$, where $\bar{g} = \det \bar{g}_{ij}$. In the absence of electromagnetic field, Eqs. (5) and (6) reduce to the classical problem, well known in the general relativity, of constructing a synchronous reference system where $g^{0m} = 0$ for $m = 1, \dots, 9$. In particular, Eq. (6) reduces to the Laplace equation $\bar{\nabla}_k \bar{\nabla}^k S = 0$ which is equivalent to the coordinate condition $\bar{R}\sqrt{\bar{g}} = \text{const.}$ along the time lines of the synchronous reference system. We may therefore consider the curvature \bar{R} as a scalar volume measure transported along the particle paths in the synchronous reference system. It is just the double role played by the scalar curvature \bar{R} , i.e. as a potential acting on the particle and as a volume measure along the particle path, which simulates in the classical framework the features of quantum mechanics. When written out in full, Eqs. (5) and (6) are a set of nonlinear partial differential equations for the unknown functions $S(q)$ and $\chi(q)$, once the metric tensor $g_{ij}(q)$ is prescribed [actually, g_{ij} it is the metric tensor defined by the Lagrangian of the “classical” top given by Eqs. (1) or (3)]. The nonlinear problem posed by Eqs. (5) and (6) may look very hard at first glance, so it is a not trivial fact that introducing the auxiliary complex scalar function

$$\psi(q) = \chi(q)^{-\frac{n-2}{2}} e^{i\frac{S(q)}{\hbar}} \quad (7)$$

and fixing γ according to

$$\gamma^2 = \frac{n-2}{4(n-1)} = \frac{2}{9}, \quad (8)$$

where $n = 10$ is the dimensionality of the configuration space, converts Eqs. (5) and (6) into the linear differential equation

$$g^{ij} \left(-i\hbar \nabla_i - \frac{e}{c} A_i \right) \left(-i\hbar \nabla_j - \frac{e}{c} A_j \right) \psi + \hbar^2 \gamma^2 R \psi = 0. \quad (9)$$

Equation (9) resembles the covariant Klein-Gordon wave-equation in the configuration space with the mass term $m^2 c^2$ replaced by the curvature potential term $\hbar^2 \gamma^2 R(q)$. But what is more surprising is that any explicit reference to the pre-potential $\chi(q)$ and to the metric \bar{g}_{ij} has been cancelled out from Eq. (9). In fact, the curvature $R(q)$ and the covariant derivatives ∇_i in Eq. (9) are calculated using the Cristoffel symbols derived from the metric g_{ij} defined by the “classical” Lagrangian L provided by Eqs. (1) or (3). As a consequence, the curvature R is constant, in our case, and is given by $R = 6/a^2$. It is also worth noting that the conserved current $j^i = \bar{g}^{ij}[\partial S/\partial q^j - (e/c)A_j]\sqrt{\bar{g}}$ can be recast in the familiar quantum

mechanical form

$$j^i = \frac{1}{2} \left[\psi^* g^{ij} \left(-i\hbar \nabla_j - \frac{e}{c} A_j \right) \psi + \psi g^{ij} \left(i\hbar \nabla_j - \frac{e}{c} A_j \right) \psi^* \right], \quad (10)$$

where once more any explicit reference to the metric \bar{g}_{ij} and to the pre-potential $\chi(q)$ has been cancelled out. The reduction of Eqs. (5) and (6) to the wave-equation (9) is the central result of this work, because it builds a bridge between the quantum and the classical worlds. Equation (9) is invariant under parity P , so we may look for solutions $\psi(q)$ which also are invariant under P . These solutions can be cast in the mode expansion form

$$\psi_{uv}(q) = D^{(u,v)}(\Lambda^{-1})_{\sigma'}^{\sigma} \psi_{\sigma}^{\sigma'}(x) + D^{(v,u)}(\Lambda^{-1})_{\dot{\sigma}'}^{\dot{\sigma}} \psi_{\dot{\sigma}}^{\dot{\sigma}'}(x) \quad (u \leq v) \quad (11)$$

where $D^{(u,v)}(\Lambda)_{\sigma'}^{\sigma}$ are the $(2u+1) \times (2v+1)$ matrices representing the Lorentz transformation $\Lambda(\theta) = \{e_a^{\mu}(\theta)\}$ in the irreducible representation labelled by the two numbers u, v given by $2u, 2v = 0, 1, 2, \dots$, and the $\psi_{\sigma}^{\sigma'}(x)$ and $\psi_{\dot{\sigma}}^{\dot{\sigma}'}(x)$ are expansion coefficients depending on the space-time coordinates x^{μ} alone. The matrices $D^{(u,v)}(\Lambda)$ and $D^{(v,u)}(\Lambda)$ depend on the Euler angles θ^{α} only, and provide conjugate representations of the Lorentz transformations². As the notation suggests, the invariance of $\psi_{uv}(q)$ under Lorentz transformations implies that $\psi_{\sigma}^{\sigma'}(x)$ and $\psi_{\dot{\sigma}}^{\dot{\sigma}'}(x)$ change as undotted and dotted contravariant spinors, respectively³. In Eq. (11) both dotted and undotted spinors appear, because we are interested in solutions $\psi_{uv}(q)$ of Eq. (9) which are invariant under parity P . Indeed, both terms on the right of Eq. (11) obey Eq. (9) separately, each one providing not parity invariant solutions to Eq. (9). In the case of spin 1/2, the spinors $\psi_{\sigma}^{\sigma'}(x)$ and $\psi_{\dot{\sigma}}^{\dot{\sigma}'}(x)$ have two components. The use of two-components spinors in place of the four-component Dirac's spinors have been extensively discussed in the literature [6]. In this paper, however, we will limit to parity invariant solutions of Eq. (9) described by four-component Dirac's spinors. Insertion of the expansion (11) into the wave-equation (9) yields to the following equation for the coefficients $\psi_{\sigma}^{\sigma'}(x)$ and $\psi_{\dot{\sigma}}^{\dot{\sigma}'}(x)$

$$\left[g^{\mu\nu} \left(-i\hbar \partial_{\mu} - \frac{e}{c} A_{\mu} \right) \left(-i\hbar \partial_{\nu} - \frac{e}{c} A_{\nu} \right) + \hbar^2 \gamma^2 R \right] \psi(x) + \Delta_J \psi(x) = 0 \quad (12)$$

² The two matrices are related by $[D^{(u,v)}(\Lambda)]^{\dagger} = [D^{(v,u)}(\Lambda)]^{-1}$.

³ The spinors $\psi_{\sigma}^{\sigma'}(x)$ and $\psi_{\dot{\sigma}}^{\dot{\sigma}'}(x)$ are invariant with respect to their lower indices, which are related to the spin component along the top axis.

where $\psi(x)$ denotes either $\psi_\sigma^{\sigma'}(x)$ or $\psi_\sigma^{\dot{\sigma}'}(x)$ and Δ_J is a $(2u+1) \times (2v+1)$ matrix depending on the space-time coordinates x^μ only, given by

$$\Delta_J = \left[\frac{\hbar}{a} \mathbf{J} - \frac{ea}{c} \mathbf{H} \right]^2 - \left[\frac{\hbar}{a} \mathbf{K} - \frac{ea}{c} \mathbf{E} \right]^2. \quad (13)$$

where \mathbf{J} and \mathbf{K} are the generators of the Lorentz group in the dotted or undotted (conjugate) representation, according if $\psi_\sigma^{\sigma'}(x)$ or $\psi_\sigma^{\dot{\sigma}'}(x)$ are considered. The connection with the spin 1/2 Dirac's theory is made by taking $(u, v) = (0, \frac{1}{2})$ in Eq. (11) so that $D^{(0,1/2)}(\Lambda) \in SL(2, C)$. Then, introducing the Dirac four-spinor $\Psi_D = \begin{pmatrix} \psi_\sigma^{\sigma'} \\ \psi_\sigma^{\dot{\sigma}'} \end{pmatrix}$ with $\sigma = \dot{\sigma}$ fixed, Eq. (12) yields

$$\begin{aligned} & \left[g^{\mu\nu} \left(-i\hbar\partial_\mu - \frac{e}{c} A_\mu \right) \left(-i\hbar\partial_\nu - \frac{e}{c} A_\nu \right) - \frac{e\hbar}{c} (\boldsymbol{\Sigma} \cdot \mathbf{H} - i\boldsymbol{\alpha} \cdot \mathbf{E}) \right] \Psi_D + \\ & + \left[\frac{e^2 a^2}{c^2} (H^2 - E^2) + \frac{3\hbar^2}{2a^2} (1 + 4\gamma^2) \right] \Psi_D = 0, \end{aligned} \quad (14)$$

where we posed $\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$, $\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}$, where $\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$ are usual Pauli's matrices. Setting $a = (\hbar/mc)\sqrt{3(1+4\gamma^2)/2}$, where m is the particle mass, and dropping the term $(ea/c)^2(H^2 - E^2) = (ea/c)^2(\frac{1}{2}F_{\mu\nu}F^{\mu\nu})$, Eq. (14) reduces to the square of Dirac's equation in its spinor representation [see, for example, Ref. 7, Eq. (32,7a)]. Equation (14), comprehensive of the electromagnetic term proportional to $F_{\mu\nu}F^{\mu\nu}$, was derived by Schulman by applying usual quantization rules to the relativistic top described by three Euler angles [8]. In his work, Schulman proposed also generalized wave equations for fields of arbitrary spin, which are equivalent to our Eqs. (12) and (13). We will refer to Schulman's paper for a detailed discussion about the physical implications of Eq. (14). However, it is worth noting that the term proportional to $F_{\mu\nu}F^{\mu\nu}$ in Eq. (14) can be cancelled out just replacing the scalar curvature \bar{R} in the Lagrangian (4) according to $\bar{R} \rightarrow \bar{R} - (ea\chi/c\hbar\gamma)^2(\frac{1}{2}F_{\mu\nu}F^{\mu\nu})$ so that Eq. (14) would reproduce the square of Dirac's equation exactly⁴. As final point we notice that the Lagrangian defined by Eq. (4), which is written in terms of the metric \bar{g}_{ij} , could have been equally written in terms of the metric g_{ij} of the original classical top. In fact, an alternative form of \bar{L}_0 is obtained by imposing on the manifold $\mathbf{M}_4 \times SO(3,1)$ the metric g_{ij} defined by Eq. (1) and changing appropriately the affine connections to the Weyl

⁴ The addition of a term proportional to $F_{\mu\nu}F^{\mu\nu}$ to the scalar curvature is a well-known feature of all Kaluza-Klein-like gauge theories.

connections Γ_{jk}^i given by

$$\Gamma_{jk}^i = -\{^i_{jk}\} + \delta_j^i \phi_k + \delta_k^i \phi_j + g_{jk} \phi^i, \quad (15)$$

where $\phi^i = g^{il} \phi_l$ and we defined the Weyl potential ϕ_i as $\phi_i = \chi^{-1} \partial \chi / \partial q^i$. Then, a straightforward calculation shows that \bar{L}_0 can be rewritten as

$$\bar{L}_0 = -\hbar \sqrt{R_W g_{ij} \frac{dq^i}{d\sigma} \frac{dq^j}{d\sigma}}. \quad (16)$$

where R_W is the Weyl scalar curvature calculated from the connections (15), viz.

$$R_W = R + 2(n-1) \nabla_k \phi^k - (n-1) \phi_k \phi^k = R + 2(n-1) \frac{\nabla_k \nabla^k \chi}{\chi} - n(n-1) \frac{\nabla_k \chi \nabla^k \chi}{\chi^2}. \quad (17)$$

The Lagrangian (16) is manifestly invariant under the conformal changes of the metric and Weyl potential $g_{ij} \rightarrow \rho g_{ij}$ and $\phi_i \rightarrow \phi_i - \rho^{-1} \partial \rho / \partial q^i$, respectively, provided the Weyl type $w(\chi)$ of the potential χ is $w(\chi) = 1/2$ (i.e. $\chi \rightarrow \rho^{1/2} \chi$). The Hamiltonian theory derived from Eq. (16) is conformally invariant too. As a consequence, our main Eqs. (5), (6) and (9) are Weyl-gauge invariant, provided the Weyl type of ψ in Eq. (9) is $w(\psi) = -(n-2)/4 = -2$. A direct calculation shows, in fact, that we may safely replace in all these equations the covariant derivatives $\bar{\nabla}_i$ with respect to the metric \bar{g}_{ij} with the co-covariant Weyl derivative D_i , which renders the Weyl-invariant character of these equations manifest⁵. The requirement of conformal invariance could have been taken as a first principle to change the Lagrangian of the classical top given by Eq. (1) into the Lagrangian of the quantum top given by Eq. (16).

In conclusion, we proved that any solution $\Psi_D(x)$ of Dirac's equations, when inserted into Eq. (11), provides also a solution to the problem posed by Eqs. (5) and (6) in the top configuration space $V_{10} = \mathcal{M}_4 \times SO(3,1)$. In this way, a one-to-one correspondence is established between the solutions of Dirac's problem and the solution of the classical Hamilton-Jacobi problem associated to the Lagrangian (4) or (16), without having recourse to quantization rules. The quantum spin 1/2 problem was converted into a classical problem, which nevertheless is more general, in the sense that it applies immediately to fields of arbitrary spin. We used a relativistic top described by six Euler angles so that the center-of-mass motion is different from the center-of-energy motion. This feature could provide

⁵ The co-covariant derivative D_i is defined by $D_i f = \nabla_i^{(\Gamma)} f - 2w(f) \phi_i f$, where f is any tensor quantity and $\nabla_i^{(\Gamma)}$ is the covariant derivative built from the Weyl connections in Eq. (15).

a simple interpretation of the spinning particle *zitterbewegung*. Finally, a Weyl invariant approach is also possible, which permits to make explicit the conformal invariance of Dirac's equation.

I am grateful to dott. Paolo Aniello for suggesting me very useful references.



* Electronic address: enrico.santamato@na.infn.it

- [1] J. Frenkel, Z. Phys. **37**, 243 (1926).
- [2] L. H. Thomas, Phil. Mag. **3**, 1 (1927).
- [3] J.-M-Souriau, *Structure des systèmes dynamiques*, Dunod Université (Dunod, Paris, 1969).
- [4] E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: a Modern Perspective* (John Wiley & Sons, New York, 1974).
- [5] H. Rund, *The Hamilton-Jacobi theory in the calculus of variations* (Robert E. Krieger, Huntington, N. Y., 1973).
- [6] L. M. Brown, Phys. Rev. **111**, 957 (1958).
- [7] L. D. Landau and E. M. Lifšitz, *Relativistic Quantum Theory* (Pergamon Press, New York, 1960).
- [8] L. S. Schulman, Nucl. Phys. **B18**, 595 (1970).